Section 2.4: The Chain Rule

In section 2.3, we learned how to find the derivative of a product or a quotient of two functions. In this section we will learn how to find the derivative of a *function of* a function. This kind of function is known as a *composite function* and is written as $f(g(x))$ or $f \circ g$.

According to the *chain rule*, the derivative of a composite function $f(g(x))$ is given by

$$\boxed{\frac{d}{dx} f[g(x)] = f'[g(x)]g'(x)}$$

In other words, one must first take the derivative of the outside function *f* with the respect to the inside function *g* and then multiply it by the derivative of the inside function *g* with respect to *x*.

This can also be written as

$$\boxed{\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}}$$

(Note how *dg* in the numerator and denominator appears to cancel out. Technically, this cannot happen because *dg* is an infinitessimally small number—see section 3.9 for more about infinitessimals. But the result works out the same if it did.)

Example: Find the deriviative of the function $f(x) = (x^2 + 3)^7$.

Solution: One could expand this expression as a polynomial using the binomial theorem and apply the power rule, but this would be very tedious to calculate. A much simpler method is to use the chain rule. Let the inside function $g(x)$ be $x^2 + 3$, and let the outside function $f(x)$ be g^7. Then, according to the chain rule,

$$\frac{df}{dx} = (7g^6)(2x)$$

Plugging in $g = x^2 + 3$, this becomes

$$7(x^2+3)^6(2x) = 14x(x^2+3)^6$$

<u>Extending the chain</u>

There is no limit to the number of times the chain rule may be applied. For example, one may find the derivative of a function $f(g(h(x)))$ using the formula

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dh}\frac{dh}{dx}$$

The meaning of "with respect to"

Since the chain rule requires you to calculate the derivative of functions with respect to multiple variables, this is probably a good time to discuss the precise meaning of the phrase "with respect to." When one takes the derivative of a function *f* with respect to a variable *x*, one is really asking the question "How rapidly is the *f* changing when *x* is varied?"

To show how the derivative depends on which variable you are taken the derivative with respect to, consider the following examples:

1) suppose that $y = f(x)$ is a function that depends on *x.* Then the derivative of *y* with respect to *x* is by definition *y'.*

$$\frac{d}{dx}y = y'$$

2) On the other hand, the derivative of *y* with respect to *y* asks "How fast does *y* change when you adjust *y*". Since the things being changed are the same, their ratio of the rates of change equals one. We may use the power rule to prove that

$$\frac{d}{dy}y = 1$$

3) Finally, if *z* is some other variable that is independent of *y,* then *y* doesn't change at all when *z* is adjusted. Hence,

$$\frac{d}{dz}y = 0$$

Suppose we have some expression involving *y*, such as y^2 where $y = f(x)$ is a function of *x.* In that case, y^2 is a composite function, so to calculate its derivative, one must use the chain rule.

$$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx} = (2y)(y') = 2yy'$$

Note the additional term *y'* would not be there if we were merely taking the derivative with respect to *y.*

Section 2.5: Implicit Differentiation

When caulculating *dy*/*dx*, one normally solves an equation for y in terms of *x,* and then differentiates. However, there are many situations in which it is more convenient to take the derivative of both sides of the equation first, and *then* solve for *dy/d*x. This procedure is known as *implicit differentiation.*

For example, consider the equation

$$y^3 + 3y^2 = 2\sin x$$

To solve this equation directly for *y* would be extremely difficult, because it would require use of the cubic equation. However, taking the derivative implicitly results in

$$3y^2y' + 6yy' = 2\cos x$$

which may be readily solved for y' using algebra

$$y' = \frac{2\cos x}{3y^2 + 6y}$$

Note that the resulting expression contains two variables: x and y. In order to calculate the numerical value of the derivative, one must know them both. This is the drawback to differentiating implicitly. On the other hand, taking the derivative implicitly often reveals symmetries or simplifications that are not apparent when the derivative is taken in the usual manner.

For example, the equation of a circle of radius r with its center at the origin is

$$x^2 + y^2 = r^2$$

Solving for y and taking the derivative in the usual way results in

$$y' = \pm \frac{x}{\sqrt{r^2 - x^2}}$$

while implicit differentiaton gives us the much simpler expression (note that since r^2 is a constant, its derivative simply disappears)

$$y' = -\frac{x}{y}$$

Section 2.6: Related Rates

It is often useful to know how one quantity changes when another is adjusted—for example how fast is the radius of a balloon is increasing when air is flowing into it at a constant rate, or how fast is the distance between two vehicles growing when they move apart from each other at a given angle and speeds. Such problems are known as *related rates* problems, and their solution involves the use of both the chain rule and implicit differentiation.

Procedure for solving a related rates problem:

1. Write an equation relating the variables whose rates are given to those whose rates you desire to find
2. If necessary, eliminate all variables except the one whose rates of change you already know or want to find out
3. Take the derivative implicitly with respect to time
4. Solve for the desired rate using algebra
5. Plug in all known quantities and simplify

It is extremely important not to plug in the values of any varying quantities before you take the derivative, because to do so will inadvertantly eliminate a source of variation.